

# STRINGS OF CONGRUENT PRIMES IN SHORT INTERVALS

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ABSTRACT. Fix  $\epsilon > 0$ , and let  $p_1 = 2, p_2 = 3, \dots$  be the sequence of all primes. We prove that if  $(q, a) = 1$  then there are infinitely many pairs  $p_r, p_{r+1}$  such that  $p_r \equiv p_{r+1} \equiv a \pmod{q}$  and  $p_{r+1} - p_r < \epsilon \log p_r$ . The proof combines the ideas of Shiu [9] and Goldston-Pintz-Yıldırım [6].

## 1. INTRODUCTION

Fix any  $\epsilon > 0$ . In 2005, Goldston, Pintz and Yıldırım proved [4, 6] that there are arbitrarily large  $x$  for which there are at least two primes in the interval  $(x, x + \epsilon \log x]$ , thus establishing the longstanding conjecture that there are infinitely many pairs of consecutive primes  $p_r, p_{r+1}$  with  $p_{r+1} - p_r < \epsilon \log p_r$ .

In [5] they extended their original argument to prove that there are arbitrarily large  $x$  for which there are at least two primes in the interval  $(x, x + \epsilon \log x]$  which are both in the arithmetic progression  $a \pmod{q}$ , provided  $(q, a) = 1$ . However one cannot deduce that these are consecutive primes for there might be a prime in-between them that is not  $\equiv a \pmod{q}$ . Hence one can only deduce that *either* there are infinitely many pairs of consecutive primes  $p_r \equiv p_{r+1} \equiv a \pmod{q}$  with  $p_{r+1} - p_r < \epsilon \log p_r$ , *or* that there are infinitely many triples of consecutive primes  $p_r, p_{r+1}, p_{r+2}$  with  $p_{r+2} - p_r < \epsilon \log p_r$ . Presumably both statements are true but one can only deduce that one of them is true, and one does not know which one, from the result in [5].

In [9], Shiu proved an old conjecture of Chowla that there are infinitely many pairs of consecutive primes  $p_r, p_{r+1}$  which are both  $\equiv a \pmod{q}$ . Indeed he was even able to extend this to  $k$  consecutive primes. In this paper we will combine the methods of Goldston-Pintz-Yıldırım and of Shiu to establish the following hybrid of those results:

**Theorem 1.1.** *Let  $q \geq 3$  and  $a$  be integers with  $(q, a) = 1$ , and fix any  $\epsilon > 0$ . There exist infinitely many pairs of consecutive primes  $p_r, p_{r+1}$  such that  $p_r \equiv p_{r+1} \equiv a \pmod{q}$  and  $p_{r+1} - p_r < \epsilon \log p_r$ .*

## 2. PRELIMINARIES

In this section we will state two key technical propositions, to be proved in sections 4 and 5. The first proposition requires some preparation. We begin by quoting the Landau-Page theorem, a proof of which can be found in [2, Chapter 14]. This theorem is used to handle problems arising from possible irregularities in the distribution of primes, hence in Bombieri-Vinogradov type theorems (see Lemma 4.2), caused by potential Siegel zeros.

**Lemma 2.1** (Landau-Page theorem). *There exists a constant  $c$  such that the following holds for any  $Y > c$ . There is at most one integer  $q_0 \leq Y$ , and at most one real primitive character  $\chi_0 \bmod q_0$ , such that*

$$L(1 - \delta, \chi_0, q_0) = 0 \quad \text{for some} \quad \delta \leq \frac{1}{3 \log Y}.$$

*If  $q_0$  exists, then  $q_0 > (\log Y)^2$ . We call  $\chi_0$  an exceptional character and  $q_0$  an exceptional modulus.*

Throughout, we fix a number  $\epsilon > 0$ , we let  $H$  be a real parameter tending monotonically to infinity, and we set  $N := \exp(H/\epsilon)$ , that is  $H = \epsilon \log N$ . If there is an exceptional modulus  $q_0 := q_0(H) \leq \exp(H/\epsilon(\log(H/\epsilon))^2) = N^{1/(\log \log N)^2}$ , let  $p_0 := p_0(H)$  be its greatest prime factor; otherwise let  $p_0 = 1$ .

For all sufficiently large  $H$ , either

$$p_0 = 1 \text{ or } p_0 \text{ is a prime with } p_0 > \log H. \tag{2.1}$$

To see this, note that all real primitive characters are products of Legendre symbols with different odd primes, and possibly either the unique real character mod 4 or one of the two primitive real characters mod 8. Thus if  $q_0$  exists it is of the form  $2^\alpha p_1 \cdots p_k$ , where  $\alpha \leq 3$  and the  $p_i$ 's are distinct odd primes. If this is the case and  $p_0 \leq \log H$ , then the prime number theorem implies  $q_0 \ll \exp((1 + o(1)) \log H) \ll \log N$ , but Lemma 2.1 states that  $q_0 > (\log N/(\log \log N)^2)^2$ .

We let  $Q := Q(H)$  be a positive integer, upon which we will impose the following conditions:

$$Q \text{ is composed only of primes } p \leq H, \tag{2.2}$$

$$Q \text{ is divisible by all primes } p \leq \log H, \tag{2.3}$$

$$Q \leq \exp(cH/(\log H)^2) \text{ for some constant } c > 0, \tag{2.4}$$

$$\text{if } p_0(H) \neq 1 \text{ then } p_0(H) \text{ does not divide } Q. \tag{2.5}$$

We let

$$\mathcal{H} := \{Qx + h_1, \dots, Qx + h_k\}, \quad h_1, \dots, h_k \in [1, H] \cap \mathbb{Z}, \quad (2.6)$$

denote a set of distinct linear forms, and we define

$$\Lambda_R(n; \mathcal{H}, j) := \frac{1}{j!} \sum'_{\substack{d|P(n; \mathcal{H}) \\ d \leq R}} \mu(d) (\log R/d)^j, \quad (2.7)$$

where  $\sum'$  denotes summation over indices coprime with  $Qp_0$ , and

$$P(n; \mathcal{H}) := (Qn + h_1) \cdots (Qn + h_k). \quad (2.8)$$

Finally, we let

$$\vartheta(n) := \begin{cases} \log n & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 2.2.** *Given  $\epsilon > 0$  and sufficiently large  $H$ , let  $N$  and  $p_0 = p_0(H)$  be as defined earlier, and let  $Q = Q(H)$  be a positive integer satisfying (2.2) – (2.5). Fix positive integers  $k$  and  $\ell$ , and let  $\mathcal{H} = \{Qx + h_1, \dots, Qx + h_k\}$  be a set of distinct linear forms with  $h_1, \dots, h_k \in [1, H] \cap \mathbb{Z}$  and  $(Q, h_1, \dots, h_k) = 1$ . Let  $h \in [1, H] \cap \mathbb{Z}$  and suppose  $(Q, h) = 1$ , and let  $R = N^{1/4-\epsilon'}$  for some  $\epsilon' \in (0, 1/4)$ . As  $H \rightarrow \infty$ , we have*

$$\frac{1}{N} \left( \frac{\phi(Q)}{Q} \right)^k \sum_{N < n \leq 2N} \Lambda_R(n; \mathcal{H}, k + \ell)^2 \sim \binom{2\ell}{\ell} \frac{(\log R)^{k+2\ell}}{(k + 2\ell)!} \quad (2.9)$$

and

$$\begin{aligned} \frac{1}{N} \left( \frac{\phi(Q)}{Q} \right)^k \sum_{N < n \leq 2N} \vartheta(Qn + h) \Lambda_R(n; \mathcal{H}, k + \ell)^2 \\ \sim \begin{cases} \frac{Q}{\phi(Q)} \binom{2\ell}{\ell} \frac{(\log R)^{k+2\ell}}{(k + 2\ell)!} & \text{if } Qx + h \notin \mathcal{H}, \\ \binom{2(\ell + 1)}{\ell + 1} \frac{(\log R)^{k+2\ell+1}}{(k + 2\ell + 1)!} & \text{if } Qx + h \in \mathcal{H}. \end{cases} \end{aligned} \quad (2.10)$$

**Proposition 2.3.** *Let  $q \geq 3$  and  $a$  be integers with  $(q, a) = 1$ , and for a given  $H$ , let  $p_0 = p_0(H)$  be as defined earlier. There is an infinite sequence of integers  $H_1 < H_2 < \dots$  such that for any  $i$ , taking  $H = H_i$ , there exists a positive integer  $Q = Q(H)$ , divisible by  $q$  and satisfying (2.2) – (2.5), such that*

$$|S| - |T| \gg_q H \left( \frac{\phi(Q)}{Q} \right), \quad (2.11)$$

where

$$\begin{aligned} S &= S(H) := \{h \in (0, H] : (Q, h) = 1 \text{ and } h \equiv a \pmod{q}\}, \\ T &= T(H) := \{h \in (0, H] : (Q, h) = 1 \text{ and } h \not\equiv a \pmod{q}\}. \end{aligned} \quad (2.12)$$

The implied constant in (2.11) depends at most on  $q$ .

### 3. PROOF OF THEOREM 1.1

Fix integers  $q \geq 3$  and  $a$  with  $(q, a) = 1$ . Recall that  $H = \epsilon \log N$ , with  $\epsilon > 0$  fixed, and  $p_0$  is the greatest prime factor of the exceptional modulus  $q_0 \leq N^{1/(\log \log N)^2}$ , if it exists, otherwise  $p_0 = 1$ . We choose  $H$ ,  $Q = Q(H)$ ,  $S = S(H)$ , and  $T = T(H)$  as in Proposition 2.3, so that  $Q$  is divisible by  $q$  and satisfies (2.2) – (2.5), and

$$\frac{Q}{\phi(Q)} \frac{|S| - |T|}{\log N} \geq c(q)\epsilon \quad (3.1)$$

for some constant  $c(q) > 0$ , depending on  $q$  at most.

We fix positive integers  $k, \ell$  (to be specified later), and we let  $\mathcal{H} = \{Qx + h_1, \dots, Qx + h_k\}$  be a set of distinct linear forms such that, for each  $i$ ,  $h_i \in [1, H] \cap a \pmod{q}$  and  $(Q, h_i) = 1$ . We let  $R = N^{1/4-\epsilon'}$  with  $0 < \epsilon' < 1/4$  (to be specified later), and we put

$$\begin{aligned} \mathcal{L} &:= \\ &\frac{1}{N} \left( \frac{\phi(Q)}{Q} \right)^k \sum_{N < n \leq 2N} \left( \sum_{h \in S} \vartheta(Qn + h) - \sum_{h \in T} \vartheta(Qn + h) - \log 3QN \right) \Lambda_R(n; \mathcal{H}, k + \ell)^2. \end{aligned}$$

We now show that if  $\mathcal{L} > 0$  for a sequence of numbers  $N$ , tending to infinity, then Theorem 1.1 follows.

Let

$$\begin{aligned} A_n &:= \{p \in (Qn, Qn + H] : p \equiv a \pmod{q}\} = \{p : p = Qn + h, h \in S\} \\ B_n &:= \{p \in (Qn, Qn + H] : p \not\equiv a \pmod{q}\} = \{p : p = Qn + h, h \in T\}. \end{aligned}$$

If  $\mathcal{L} > 0$ , then there is some  $n \in (N, 2N]$  such that

$$|A_n| \log(Qn + H) \geq \sum_{h \in S} \vartheta(Qn + h) > \sum_{h \in T} \vartheta(Qn + h) + \log 3QN \geq |B_n| \log Qn + \log 3QN.$$

Now

$$|A_n| \log(1 + H/Qn) \leq |A_n| H/Qn \leq H^2/QN < \log(3/2)$$

if  $N$  is sufficiently large, and so

$$\log(3/2) + (|A_n| - |B_n|) \log Qn > \log 3QN$$

and hence, as  $n \leq 2N$ ,  $|A_n| - |B_n| > 1$ . But as these are integers,  $|A_n| \geq |B_n| + 2$ , and so, by the pigeonhole principle,  $A_n$  contains a pair of consecutive primes  $p_r, p_{r+1}$ . These primes satisfy  $p_{r+1} - p_r < H < \epsilon \log QN < \epsilon \log p_r$ .

Now, by our choice of  $\mathcal{H}$ , a straightforward application of Proposition 2.2 yields

$$\begin{aligned} \mathcal{L} &= \binom{2\ell}{\ell} \frac{(\log R)^{k+2\ell}}{(k+2\ell)!} \\ &\times \left\{ \frac{Q}{\phi(Q)} \sum_{\substack{h \in S \\ Qx+h \notin \mathcal{H}}} 1 + \frac{2(2\ell+1)}{\ell+1} \frac{\log R}{k+2\ell+1} \sum_{\substack{h \in S \\ Qx+h \in \mathcal{H}}} 1 - \frac{Q}{\phi(Q)} \sum_{h \in T} 1 - (1+o(1)) \log 3QN \right\}. \end{aligned}$$

We have

$$\sum_{\substack{h \in S \\ Qx+h \in \mathcal{H}}} 1 = k, \quad \sum_{\substack{h \in S \\ Qx+h \notin \mathcal{H}}} 1 = |S| - k,$$

$\log R = (1/4 - \epsilon') \log N$ , and  $\log 3QN \sim \log N$  by (2.4), therefore

$$\begin{aligned} \mathcal{L} &= \binom{2\ell}{\ell} \frac{(\log R)^{k+2\ell}}{(k+2\ell)!} \log N \\ &\times \left\{ \frac{Q}{\phi(Q)} \frac{|S| - |T|}{\log N} + \frac{2(2\ell+1)}{\ell+1} \frac{k}{k+2\ell+1} \left( \frac{1}{4} - \epsilon' \right) - (1+o(1)) \right\}. \end{aligned}$$

We have written  $o(1)$  for  $kQ/(\phi(Q) \log N)$ , because  $Q/\phi(Q) \ll \log \log Q \ll \log \log N$ .

By choosing  $\ell = \lfloor \sqrt{k} \rfloor$  and  $k$  sufficiently large, the bracketed expression  $\{\cdots\}$  above is, by (3.1),

$$\geq c(q)\epsilon + 1 - 5\epsilon' - (1+o(1)) = c(q)\epsilon - 5\epsilon' - o(1).$$

By choosing  $\epsilon' = c(q)\epsilon/10$  (we may assume that  $\epsilon$  is small enough so that  $\epsilon' < 1/4$ ), we deduce that

$$\mathcal{L} \gg_k c(q)\epsilon (\log N)^{k+2\ell+1} \tag{3.2}$$

holds if  $N$  is sufficiently large. By Proposition 2.3, we may choose  $H$ , equivalently  $N$ , from a sequence of numbers tending to infinity, and Theorem 1.1 follows.

#### 4. PROOF OF PROPOSITION 2.2

The estimates (2.9) and (2.10) of Proposition 2.2 are essentially the same as estimates already in the literature, so we will only outline a proof of each of them, referring to [3] and [5] for details.

Let  $Q = Q(H)$  satisfy (2.2) and (2.3). For a set of distinct linear forms  $\mathcal{H}$ , as in (2.6), and positive integers  $d$ , we define

$$\Omega(d) = \Omega(d; \mathcal{H}) := \{n \bmod d : P(n; \mathcal{H}) \equiv 0 \bmod d\},$$

where  $P(n; \mathcal{H})$  is as in (2.8). A Chinese remainder theorem argument shows that  $n \bmod d \in \Omega(d)$  if and only if  $p^r \parallel P(n; \mathcal{H})$  for every  $p^r \parallel d$ , and so  $|\Omega(d)|$  defines a multiplicative function of  $d$ . Thus, if we define

$$\lambda_R(d; j) := \begin{cases} \frac{1}{j!} \mu(d) (\log R/d)^j & \text{if } d \leq R, \\ 0 & \text{if } d > R, \end{cases} \quad (4.1)$$

we see from (2.7) that

$$\Lambda_R(n; \mathcal{H}, j) := \frac{1}{j!} \sum'_{\substack{d | P(n; \mathcal{H}) \\ d \leq R}} \mu(d) (\log R/d)^j = \sum'_{\substack{n \bmod d \\ \in \Omega(d)}} \lambda_R(d; j). \quad (4.2)$$

We call  $\mathcal{H}$  admissible if  $|\Omega(p)| < p$  for all  $p$ , and one can prove that this is equivalent to  $\mathfrak{S}(\mathcal{H}) \neq 0$ , where

$$\mathfrak{S}(\mathcal{H}) := \prod_p \left(1 - \frac{|\Omega(p)|}{p}\right) \left(1 - \frac{1}{p}\right)^{-k}$$

is the singular series for  $\mathcal{H}$ .

**Lemma 4.1.** *Let  $H$  be a real number, let  $Q = Q(H)$  be a positive integer satisfying (2.2) and (2.3), and let  $\mathcal{H}$  be as in (2.6), with  $k$  fixed. We have*

$$|\Omega(p)| = k \quad \text{for all } p > H. \quad (4.3)$$

*For  $k \leq \log H$ ,  $\mathcal{H}$  is admissible if and only if  $(Q, h_1 \cdots h_k) = 1$ . Moreover, as  $H \rightarrow \infty$ , for  $(Q, h_1 \cdots h_k) = 1$  we have*

$$\mathfrak{S}(\mathcal{H}) \sim \left( \frac{Q}{\phi(Q)} \right)^k. \quad (4.4)$$

*Proof.* For primes  $p$  that do not divide  $Q$ , we have

$$\Omega(p) = \{-h_1 Q^{-1}, \dots, -h_k Q^{-1}\} \bmod p,$$

and hence  $1 \leq |\Omega(p)| \leq \min(k, p)$ . For such  $p$ , we have  $|\Omega(p)| = k$  if and only if the  $-h_i Q^{-1}$  are all distinct modulo  $p$ , that is if and only if  $p \nmid \Delta$ , where

$$\Delta = \Delta(\mathcal{H}) := \prod_{1 \leq i < j \leq k} |h_i - h_j|.$$

By (2.2),  $p > H$  implies  $p \nmid Q$ , and since  $1 \leq |h_i - h_j| \leq H$  for every  $i, j$ ,  $p > H$  also implies  $p \nmid \Delta$ , and hence  $|\Omega(p)| = k$ . We have established (4.3).

If some prime  $p$  divides  $(Q, h_1 \cdots h_k)$ , then  $P(n; \mathcal{H}) \equiv h_1 \cdots h_k \equiv 0 \pmod{p}$  for every  $n \pmod{p}$ , hence  $|\Omega(p)| = p$ , and so  $\mathcal{H}$  is not admissible if  $(Q, h_1 \cdots h_k) \neq 1$ . If  $(Q, h_1 \cdots h_k) = 1$ , then  $P(n; \mathcal{H}) \equiv h_1 \cdots h_k \not\equiv 0 \pmod{p}$ , and hence  $|\Omega(p)| = 0$ , for every  $p$  dividing  $Q$ . For every other  $p$  we have  $1 \leq |\Omega(p)| \leq \min(k, p)$ . Then for  $k \leq \log H$  and  $p \nmid Q$ , we have  $1 \leq |\Omega(p)| \leq k \leq \log H < p$  by (2.3), hence  $\mathcal{H}$  is admissible.

Now assume  $H$  is large enough so that  $\log H \geq 2k$ , and suppose  $(Q, h_1 \cdots h_k) = 1$ . Then for (4.4), since  $|\Omega(p)| = 0$  if  $p \mid Q$ , it suffices to show that

$$\mathfrak{S}'(\mathcal{H}) := \prod_{p \nmid Q} \left(1 - \frac{|\Omega(p)|}{p}\right) \left(1 - \frac{1}{p}\right)^{-k} \sim 1 \quad (4.5)$$

as  $H$  tends to infinity. We break  $\mathfrak{S}'(\mathcal{H})$  into two products according as  $p \mid \Delta$  or  $p \nmid \Delta$ , and use the fact that  $|\Omega(p)| = k$  for  $p \nmid Q\Delta$ :

$$\begin{aligned} \mathfrak{S}'(\mathcal{H}) &= \prod_{p \nmid Q} \left(1 - \frac{k}{p}\right) \left(1 + \frac{k - |\Omega(p)|}{p - k}\right) \left(1 - \frac{1}{p}\right)^{-k} \\ &= \prod_{p \nmid Q} \left(1 - \frac{k}{p}\right) \left(1 - \frac{1}{p}\right)^{-k} \prod_{\substack{p \nmid Q \\ p \mid \Delta}} \left(1 + \frac{k - |\Omega(p)|}{p - k}\right). \end{aligned} \quad (4.6)$$

In this product  $p - k \neq 0$  because, by (2.3),  $p \nmid Q$  implies  $p > \log H \geq 2k$ . For the same reason, the logarithm of the first product of the last line of (4.6) is

$$\sum_{p \nmid Q} \left\{ \left( -\frac{k}{p} - \frac{k^2}{2p^2} - \cdots \right) - k \left( -\frac{1}{p} - \frac{1}{2p^2} - \cdots \right) \right\} \ll k^2 \sum_{p > \log H} \frac{1}{p^2} \ll \frac{k^2}{\log H \log \log H}.$$

For the second product, note that since  $k/\log H \leq 1/2$ , we have

$$0 < \frac{k - |\Omega(p)|}{p - k} \leq \frac{k}{p - k} \leq \frac{2k}{p} < 1.$$

Hence the logarithm of the second product is

$$\leq \sum_{\substack{p \mid \Delta \\ p > \log H}} \log \left(1 + \frac{k - |\Omega(p)|}{p - k}\right) \ll \sum_{\substack{p \mid \Delta \\ p > \log H}} \frac{k}{p} \ll \frac{k}{\log H} \sum_{p \mid \Delta} 1 \ll \frac{k \log \Delta}{\log H \log \log \Delta} \ll \frac{k^3}{\log \log H}$$

by the prime number theorem, because  $\Delta \leq H^{\binom{k}{2}}$ . Exponentiating and letting  $H$  tend to infinity yields (4.5).  $\square$

We now assume all of the hypotheses of Proposition 2.2. The proof of (2.9) is almost identical to the proof of Lemma 1 of [3], the only difference being that primes  $p \mid Qp_0$  are excluded from the representation of  $F(s_1, s_2; \Omega)$ , where

$$\begin{aligned} F(s_1, s_2; \Omega) &:= \sum'_{d_1, d_2} \mu(d_1) \mu(d_2) \frac{|\Omega([d_1, d_2])|}{[d_1, d_2] d_1^{s_1} d_2^{s_2}} \\ &= \prod_{p \nmid Qp_0} \left( 1 - \frac{|\Omega(p)|}{p} \left( \frac{1}{p^{s_1}} + \frac{1}{p^{s_2}} - \frac{1}{p^{s_1+s_2}} \right) \right) \end{aligned}$$

in the region of absolute convergence. Since  $|\Omega(p)| = k$  for  $p > H$  by (4.3), we put

$$G(s_1, s_2; \Omega) := F(s_1, s_2; \Omega) \left( \frac{\zeta(s_1+1)\zeta(s_2+1)}{\zeta(s_1+s_2+1)} \right)^k.$$

In the proof of Lemma 1 of [3],  $G(0, 0; \Omega) = \mathfrak{S}(\mathcal{H})$ , but in our situation, we have

$$G(0, 0; \Omega) = \prod_{p \nmid Qp_0} \left( 1 - \frac{|\Omega(p)|}{p} \right) \prod_p \left( 1 - \frac{1}{p} \right)^{-k} = \mathfrak{S}(\mathcal{H}) \prod_{p \mid p_0} \left( 1 - \frac{|\Omega(p)|}{p} \right)^{-1},$$

because  $(Q, p_0) = 1$  and  $|\Omega(p)| = 0$  if  $p \mid Q$ . The last product is  $\sim 1$  by (2.1). Now applying (4.4), and proceeding as in the proof of Lemma 1 of [3], (2.9) is established.

The proof of (2.10) follows that of Lemma 2 of [3] very closely: there is one important difference concerning the error

$$E^*(N, q) := \max_{x \leq N} \max_{(a, q)=1} \left| \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p - \frac{x}{\phi(q)} \right|.$$

The usual Bombieri-Vinogradov theorem will not suffice here, but the next lemma, which is Lemma 2 of [5], will.

**Lemma 4.2.** *Let  $Q$  be an integer and  $Y, M$  be numbers such that*

$$Q^2 \leq Y \leq M, \quad \exp\left(2\sqrt{\log M}\right) \leq Y. \quad (4.7)$$

*If there is an exceptional modulus  $q_0 \leq Y$ , suppose  $p_0 \nmid Q$  for some  $p_0 \mid q_0$ ; otherwise, let  $p_0 = 1$ . If*

$$R^* := M^{1/2} Q^{-3} \exp\left(-\sqrt{\log M}\right), \quad (4.8)$$



then we have, with explicitly calculable positive constants  $c_1$  and  $c_2$ ,

$$\sum_{\substack{D \leq R^* \\ (D, Qp_0)=1}} E^*(M, QD) \leq c_1 \frac{M}{Q} \exp \left( -\frac{c_2 \log M}{\log Y} \right). \quad (4.9)$$

By (2.2) – (2.5), we see that (4.7) is satisfied with

$$Y = \exp \left( 2cH/(\log H)^2 \right) = N^{2c\epsilon(1+o(1))/(\log \log N)^2},$$

and  $M = 3QN$ . We also have

$$R^2 = N^{1/2-2\epsilon'} \leq R^* = (3QN)^{1/2} Q^{-3} \exp \left( -\sqrt{\log 3QN} \right),$$

for all sufficiently large  $N$ , and

$$c_2 \log M / \log Y = c_2(1+o(1)) \log N / \log Y = c_2(1+o(1))(\log \log N)^2 / 2c\epsilon.$$

Letting  $c_3 = c_2/12c\epsilon$  and putting this into (4.9), we deduce from Lemma 4.2 that

$$\sum'_{D \leq R^2} E^*(3QN, QD) \ll N(\log N)^{-5c_3 \log \log N} \quad (4.10)$$

for all sufficiently large  $N$ .

Now, abbreviating  $\lambda_R(d; k + \ell)$  to  $\lambda_d$ , by (4.2) we have

$$\begin{aligned} \sum_{N < n \leq 2N} \vartheta(Qn + h) \Lambda_R(n; \mathcal{H}, k + \ell)^2 &= \sum'_{d_1, d_2} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{N < n \leq 2N \\ [d_1, d_2] | P(n; \mathcal{H})}} \vartheta(Qn + h) \\ &= \sum'_{d_1, d_2} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{m \bmod [d_1, d_2] \\ \in \Omega([d_1, d_2])}} \sum_{\substack{QN+h < p \leq 2QN+h \\ p \equiv h \bmod Q \\ p \equiv Qm+h \bmod [d_1, d_2]}} \log p. \end{aligned} \quad (4.11)$$

We may assume  $(Qm + h, [d_1, d_2]) = (Q, [d_1, d_2]) = 1$  in the last sum, so we define

$$\Omega^*(d) := \Omega(d) \setminus \{m \bmod d : (Qm + h, d) \neq 1\}.$$

For  $d_1, d_2$  with  $(Q, [d_1, d_2]) = 1$  and  $m \bmod [d_1, d_2] \in \Omega^*([d_1, d_2])$ , we let  $h_m \bmod Q[d_1, d_2]$  be the unique congruence class mod  $Q[d_1, d_2]$  satisfying  $h_m \equiv h \bmod Q$  and  $h_m \equiv Qm + h \bmod [d_1, d_2]$ . Thus, the last sum in (4.11) is equal to

$$\sum_{\substack{QN+h < p \leq 2QN+h \\ p \equiv h_m \bmod Q[d_1, d_2]}} \log p = \frac{2QN + h}{\phi(Q[d_1, d_2])} - \frac{QN + h}{\phi(Q[d_1, d_2])} + O(E^*(3QN, Q[d_1, d_2])),$$

and (4.11) becomes

$$\frac{QN}{\phi(Q)} \mathcal{T}^* + O(\mathcal{E}^*), \quad (4.12)$$

with

$$\mathcal{T}^* := \sum'_{d_1, d_2} \frac{\lambda_{d_1} \lambda_{d_2} |\Omega^*([d_1, d_2])|}{\phi([d_1, d_2])}, \quad \mathcal{E}^* := \sum'_{d_1, d_2} |\lambda_{d_1} \lambda_{d_2}| |\Omega^*([d_1, d_2])| E^*(3QN, Q[d_1, d_2]).$$

Now from the definition (4.1) it is clear that  $|\lambda_d| \leq (\log R)^{k+\ell}$ . Also, as we saw in the beginning of the proof of Lemma 4.1, since  $(Q, h_1 \cdots h_k) = 1$  we have  $|\Omega(p)| \leq k$  for all  $p$ , and so  $|\Omega^*(d)| \leq |\Omega(d)| \leq k^{\omega(d)}$  for squarefree  $d$ . Thus

$$\begin{aligned} \mathcal{E}^* &\leq (\log R)^{2(k+\ell)} \sum'_{D \leq R^2} \mu^2(D) k^{\omega(D)} E^*(3QN, QD) \sum_{[d_1, d_2]=D} 1 \\ &= (\log R)^{2(k+\ell)} \sum'_{D \leq R^2} \mu^2(D) (3k)^{\omega(D)} E^*(3QN, QD). \end{aligned}$$

By the trivial inequality

$$E^*(3QN, QD) \ll \frac{QN \log QN}{QD} \ll \frac{N \log N}{D},$$

and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum'_{D \leq R^2} \mu^2(D) (3k)^{\omega(D)} E^*(3QN, QD) \\ \ll \left( N \log N \sum'_{D \leq R^2} \frac{\mu^2(D) (3k)^{2\omega(D)}}{D} \right)^{1/2} \left( \sum'_{D \leq R^2} E^*(3QN, QD) \right)^{1/2}. \end{aligned}$$

For positive integers  $\kappa$ , we have

$$\sum_{D \leq R^2} \frac{\mu^2(D) \kappa^{\omega(D)}}{D} = \sum_{d \cdots d_\kappa \leq R^2} \frac{\mu^2(d_1) \cdots \mu^2(d_\kappa)}{d_1 \cdots d_\kappa} \ll (\log R^2)^\kappa \ll (\log N)^\kappa,$$

so combining and applying (4.10) yields

$$\mathcal{E}^* \ll N \frac{(\log N)^{2(k+\ell)+(3k)^2/2+1/2}}{(\log N)^{-2c_3 \log \log N}} \leq N (\log N)^{-c_3 \log \log N}. \quad (4.13)$$

We will now evaluate  $\mathcal{T}^*$ , assuming first that  $Qx + h \notin \mathcal{H}$ . Let  $\mathcal{H}^+ = \mathcal{H} \cup \{Qx + h\}$  and observe that for  $p \nmid Q$ ,

$$|\Omega^*(p)| = |\Omega(p; \mathcal{H}^+)| - 1 := |\Omega^+(p)| - 1.$$

As with  $|\Omega(d)|$ , a Chinese remainder theorem argument shows that  $|\Omega^*(d)|$  defines a multiplicative function of  $d$ . Thus

$$|\Omega^*([d_1, d_2])| = \prod_{p|[d_1, d_2]} (|\Omega^+(p)| - 1),$$

provided  $[d_1, d_2]$  is squarefree and  $(Q, [d_1, d_2]) = 1$ , as is the case for  $d_1, d_2$  appearing in the sum defining  $\mathcal{T}^*$ .

We now proceed as in the proof of Lemma 2 of [3]: again, the only modification necessary is to  $G(0, 0; \Omega^+)$ . First note that

$$\mathfrak{S}(\mathcal{H}^+) = \prod_p \left( \frac{p - |\Omega^+(p)|}{p} \right) \left( \frac{p}{p-1} \right) \left( 1 - \frac{1}{p} \right)^{-k} = \prod_p \left( 1 - \frac{|\Omega^+(p)| - 1}{p-1} \right) \left( 1 - \frac{1}{p} \right)^{-k}.$$

By (4.3),  $|\Omega^+(p)| = |\mathcal{H}^+| = k+1$  for  $p > H$ , and if

$$G(s_1, s_2; \Omega^+) := \prod_{p \nmid Qp_0} \left( 1 - \frac{|\Omega^+(p)| - 1}{p-1} \left( \frac{1}{p^{s_1}} + \frac{1}{p^{s_1}} - \frac{1}{p^{s_1+s_2}} \right) \right) \cdot \left( \frac{\zeta(s_1+1)\zeta(s_2+1)}{\zeta(s_1+s_2+1)} \right)^k,$$

then

$$\begin{aligned} G(0, 0; \Omega^+) &= \prod_{p \nmid Qp_0} \left( 1 - \frac{|\Omega^+(p)| - 1}{p-1} \right) \prod_p \left( 1 - \frac{1}{p} \right)^{-k} \\ &= \mathfrak{S}(\mathcal{H}^+) \prod_{p \mid Q} \left( 1 + \frac{1}{p-1} \right)^{-1} \prod_{p \mid p_0} \left( 1 - \frac{|\Omega^+(p)| - 1}{p-1} \right)^{-1} \\ &\sim \left( \frac{Q}{\phi(Q)} \right)^k, \end{aligned}$$

by Lemma 4.1 and (2.1). Therefore

$$\mathcal{T}^* \sim \left( \frac{Q}{\phi(Q)} \right)^k \binom{2\ell}{\ell} \frac{(\log R)^{k+2\ell}}{(k+2\ell)!}. \quad (4.14)$$

We remark that since  $(Q, h) = (Q, h_1 \cdots h_k) = 1$ ,  $\mathcal{H}^+$  is admissible (for all sufficiently large  $N$ ) by Lemma 4.1, so we do not have to consider the other case as in the proof of Lemma 2 in [3]. Combining (4.14) with (4.13) and (4.12) yields the first case of (2.10). For the case  $Qx + h \in \mathcal{H}$ , we observe that, similarly to (2.2) of [3], we have

$$\sum_{N < n \leq 2N} \vartheta(Qn + h) \Lambda_R(n; \mathcal{H}, k + \ell)^2 = \sum_{N < n \leq 2N} \vartheta(Qn + h) \Lambda_R(n; \mathcal{H} \setminus \{Qx + h\}, k + \ell)^2,$$

so the above evaluation applies with the translation  $k \mapsto k-1$ ,  $\ell \mapsto \ell+1$  to (4.14).

## 5. PROOF OF PROPOSITION 2.3

**5.1. Auxiliary lemmas.** To prove Proposition 2.3, we will use the following lemmas.

**Lemma 5.1.** *Fix integers  $q$  and  $a$  with  $(q, a) = 1$ . There is a constant  $c(q, a) > 0$ , depending only on  $q$  and  $a$ , such that*

$$\prod_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \left(1 - \frac{1}{p}\right) \sim \frac{c(q, a)}{(\log x)^{1/\phi(q)}}$$

as  $x \rightarrow \infty$ .

*Proof.* This follows from the prime number theorem for arithmetic progressions. For a more precise estimate, with the constant  $c(q, a)$  given explicitly, see [10, Theorem 1].  $\square$

**Lemma 5.2.** *Let  $\mathcal{S}(x)$  denote the set of positive integers which are  $\leq x$  and composed only of primes  $p \equiv 1 \pmod{q}$ . There is a constant  $c(q) > 0$ , depending only on  $q$ , such that*

$$|\mathcal{S}(x)| = \left( c(q) + O\left(\frac{1}{\log x}\right) \right) \frac{x}{\log x} (\log x)^{1/\phi(q)}.$$

*Proof.* See [9, Lemma 3], in which the constant  $c(q)$  is given explicitly.  $\square$

The next lemma concerns  $\Psi(x, y)$ , the number of positive integers which are  $\leq x$  and free of prime factors  $> y$  ( $y$ -smooth numbers). The ratio  $\Psi(x, y)/x$  depends essentially on  $u = \log x / \log y$ , and for  $u$  in a certain range is approximated by  $\rho(u)$ , where  $\rho(u)$  is the Dickman-de Bruijn  $\rho$ -function, defined as the continuous solution to

$$\rho(u) := \begin{cases} 1 & 0 \leq u \leq 1, \\ \frac{1}{u} \int_{u-1}^u \rho(t) dt & u > 1. \end{cases} \quad (5.1)$$

**Lemma 5.3.** *The estimate*

$$\frac{\Psi(y^u, y)}{y^u} = \rho(u) \left( 1 + O\left(\frac{\log(u+2)}{\log y}\right) \right) \quad (5.2)$$

*holds uniformly in the range*

$$y \geq 3, \quad 1 \leq u \leq \exp((\log y)^{3/5-\delta}), \quad (5.3)$$

*where  $\delta$  is any fixed positive number. The estimate*

$$\rho(u) = \exp(-u \log u - u \log \log u + O(u)) \quad (5.4)$$

*holds for  $u > 3$ , and*

$$\frac{\Psi(y^u, y)}{y^u} = \exp(-u \log u - u \log \log u + O(u)) \quad (5.5)$$

holds uniformly in the range

$$3 < u \leq y^{1-\delta}. \quad (5.6)$$

Finally, as  $y \rightarrow \infty$ ,

$$\frac{\Psi(y, (\log y)^A)}{y} = \frac{1}{y^{1/A+o(1)}} \quad (5.7)$$

holds for any fixed number  $A > 1$ .

*Proof.* We refer to the survey article of Granville [7]. The asymptotic (5.2) was shown to hold for the range (5.3) by Hildebrand [8]: see [7, (1.8), (1.10)]. Hildebrand [8] also established that the less precise estimate

$$\frac{\Psi(y^u, y)}{y^u} = \rho(u) \exp(O_\delta(u \exp(-(\log u)^{3/5-\delta})))$$

holds, for any fixed number  $\delta > 0$ , in the wider range (5.6). (See displayed formulas [7, (1.11), (1.13)].) That (5.5) holds in the same range can be deduced from (5.4). (The estimate (5.5) is less precise, but sufficient for our purposes.) For the estimate (5.7), see [7, (1.14)].

The value of the Dickman-de Bruijn  $\rho$ -function is discussed in [7, 3.7 – 3.9], and (5.4) was proved by de Bruijn in [1].  $\square$

**Lemma 5.4.** *Let  $\mathcal{P}$  be a subset of the primes. As  $y \rightarrow \infty$ , the estimate*

$$\prod_{\substack{p \leq y \\ p \in \mathcal{P}}} \left(1 - \frac{1}{p}\right) \sum_{\substack{n > y^u \\ p|n \Rightarrow p \leq y \\ p \in \mathcal{P}}} \frac{1}{n} \leq (1 + o(1))e^{-\gamma} \int_u^\infty \rho(v) dv. \quad (5.8)$$

holds uniformly for  $u$  satisfying

$$u \geq 1, \quad u = \exp((\log y)^{3/5-\delta}), \quad (5.9)$$

where  $\delta$  is any fixed positive number.

*Proof.* Define

$$\varrho(x, y; \mathcal{P}) := \prod_{\substack{p \leq y \\ p \in \mathcal{P}}} \left(1 - \frac{1}{p}\right) \sum_{\substack{n \leq x \\ p|n \Rightarrow p \leq y \\ p \in \mathcal{P}}} \frac{1}{n}.$$

If  $\ell \leq y$  is prime, then

$$\varrho(x, y; \mathcal{P}) = \prod_{\substack{p \leq y \\ p \in \mathcal{P} \cup \{\ell\}}} \left(1 - \frac{1}{p}\right) \cdot \left(1 - \frac{1}{\ell}\right)^{-1} \sum_{\substack{n \leq x \\ p|n \Rightarrow p \leq y \\ p \in \mathcal{P}}} \frac{1}{n}.$$

Now

$$\left(1 - \frac{1}{\ell}\right)^{-1} \sum_{\substack{n \leq x \\ p|n \Rightarrow p \leq y \\ p \in \mathcal{P}}} \frac{1}{n} = \left(1 + \frac{1}{\ell} + \frac{1}{\ell^2} + \cdots\right) \sum_{\substack{n \leq x \\ p|n \Rightarrow p \leq y \\ p \in \mathcal{P}}} \frac{1}{n} \geq \sum_{\substack{m \leq x \\ p|m \Rightarrow p \leq y \\ p \in \mathcal{P} \cup \{\ell\}}} \frac{1}{m},$$

because every  $m$  appearing in the last sum may be written as  $n\ell^\alpha$  for some  $\alpha \geq 0$  and some  $n$  appearing in the second last sum. Hence,

$$\varrho(x, y; \mathcal{P}) \geq \varrho(x, y; \mathcal{P} \cup \{\ell\}),$$

and applying this inequality repeatedly, we obtain

$$\varrho(x, y; \mathcal{P}) \geq \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sum_{\substack{n \leq x \\ p|n \Rightarrow p \leq y}} \frac{1}{n}.$$

Subtracting both sides from  $\varrho(\infty, y; \mathcal{P}) = 1 = \varrho(\infty, y; \{p \leq y\})$ , we deduce that

$$\prod_{\substack{p \leq y \\ p \in \mathcal{P}}} \left(1 - \frac{1}{p}\right) \sum_{\substack{n > x \\ p|n \Rightarrow p \leq y \\ p \in \mathcal{P}}} \frac{1}{n} \leq \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sum_{\substack{n > x \\ p|n \Rightarrow p \leq y}} \frac{1}{n}. \quad (5.10)$$

By partial summation,

$$\sum_{\substack{n > x \\ p|n \Rightarrow p \leq y}} \frac{1}{n} = \int_x^\infty \frac{d\Psi(t, y)}{t} = -\frac{\Psi(x, y)}{x} + \int_x^\infty \frac{\Psi(t, y)}{t^2} dt \leq \int_x^\infty \frac{\Psi(t, y)}{t^2} dt. \quad (5.11)$$

Now we assume  $x = y^u$ , with  $u$  satisfying (5.9) and  $y$  tending to infinity. We will divide the range of the last integral in (5.11) into three parts. First of all, fix any  $\epsilon \in (0, 1)$  and suppose  $t \geq \exp(y^\epsilon)$ , that is  $y \leq (\log t)^{1/\epsilon}$ . By (5.7) we have

$$\frac{\Psi(t, y)}{t^2} \leq \frac{\Psi(t, (\log t)^{1/\epsilon})}{t^2} = \frac{1}{t^{1+\epsilon+o(1)}}$$

as  $t$ , and hence as  $y$ , tends to infinity. Thus, we may suppose  $y$  is large enough so that  $\Psi(t, y)/t^2 \leq 1/t^{1+\epsilon/2}$ , say, and

$$\int_{\exp(y^\epsilon)}^\infty \frac{\Psi(t, y)}{t^2} dt \leq \int_{\exp(y^\epsilon)}^\infty \frac{dt}{t^{1+\epsilon/2}} = \frac{2}{\epsilon \exp(\epsilon y^\epsilon/2)}. \quad (5.12)$$

For the range  $x \leq t \leq \exp(y^\epsilon)$ , the substitution  $t = y^v$  yields

$$\int_x^{\exp(y^\epsilon)} \frac{\Psi(t, y)}{t^2} dt = \log y \int_u^{y^\epsilon / \log y} \frac{\Psi(y^v, y)}{y^v} dv. \quad (5.13)$$

Next, we let  $u_1 = 2 \exp((\log y)^{3/5-\delta})$ , and for  $u_1 \leq v \leq y^\epsilon$ , we use the estimate (5.5):

$$\frac{\Psi(y^v, y)}{y^v} = \exp(-v \log v - v \log \log v + O(v)) \leq \frac{1}{v^v},$$

where the last inequality holds for all sufficiently large  $v$ , hence for all sufficiently large  $y$ . Thus

$$\int_{u_1}^{y^\epsilon / \log y} \frac{\Psi(y^v, y)}{y^v} dv \leq \int_{u_1}^\infty \frac{dv}{v^v} \ll \frac{1}{u_1^{u_1}} \quad (5.14)$$

for all sufficiently large  $y$ .

For  $u \leq v \leq u_1$ , we use the estimate (5.2):

$$\begin{aligned} \int_u^{u_1} \frac{\Psi(y^v, y)}{y^v} dv &= \int_u^{u_1} \rho(v) \left( 1 + O\left(\frac{\log(v+2)}{\log y}\right) \right) dv \\ &= (1 + o(1)) \int_u^\infty \rho(v) dv - (1 + o(1)) \int_{u_1}^\infty \rho(v) dv. \end{aligned} \quad (5.15)$$

By (5.4) we have, similarly to (5.14), the estimate

$$\int_{u_1}^\infty \rho(v) dv \leq \int_{u_1}^\infty \frac{dv}{v^v} \ll \frac{1}{u_1^{u_1}} \quad (5.16)$$

for all sufficiently large  $y$ .

Combining (5.11) – (5.16), we see that

$$\int_x^\infty \frac{\Psi(t, y)}{t^2} dt = (1 + o(1)) \log y \int_u^\infty \rho(v) dv + O(u_1^{-u_1} \log y) \quad (5.17)$$

for all sufficiently large  $y$ . Now by definition (5.1),

$$\int_u^\infty \rho(v) dv \geq \int_u^{u+1} \rho(v) dv = (u+1)\rho(u+1),$$

and by (5.4),  $u_1^{-u_1} = o((u+1)\rho(u+1))$  as  $u_1 \geq 2u$ , and  $u_1$  tends to infinity with  $y$ . Therefore, combining (5.17) with (5.11) in fact gives

$$\sum_{\substack{n > y^u \\ p|n \Rightarrow p \leq y}} \frac{1}{n} \leq (1 + o(1)) \log y \int_u^\infty \rho(v) dv \quad (5.18)$$

as  $y \rightarrow \infty$ , for  $u$  in the range (5.9). Finally, combining (5.18) with (5.10) and applying Mertens' theorem, we obtain (5.8).  $\square$

**5.2. The proof of Proposition 2.3.** We are now ready to define  $Q$  explicitly. The construction is modelled on that of Shiu's [9]. For the rest of this section we let  $q \geq 3$  and  $a$  be

integers with  $(q, a) = 1$ . If  $a \equiv 1 \pmod{q}$ , let

$$\mathcal{P}(H) := \{p \leq \log H : p \equiv 1 \pmod{q}\} \cup \{p \leq H/(\log H)^2 : p \not\equiv 1 \pmod{q}\},$$

otherwise let

$$\begin{aligned} \mathcal{P}(H) &:= \{p \leq \log H : p \equiv 1 \pmod{q}\} \cup \{p \leq H/(\log H)^2 : p \not\equiv 1, a \pmod{q}\} \\ &\cup \{t(H) \leq p \leq H/(\log H)^2 : p \equiv 1 \pmod{q}\} \cup \{p \leq H/t(H) : p \equiv a \pmod{q}\}, \end{aligned}$$

with

$$t(H) := \exp \left( \frac{\log H \log \log \log H}{2 \log \log H} \right),$$

and put

$$\tilde{Q}(H) := q \prod_{p \in \mathcal{P}(H)} p, \quad Q = Q(H) := q \prod_{\substack{p \in \mathcal{P}(H) \\ p \neq p_0}} p. \quad (5.19)$$

We check that (2.2) – (2.5) are indeed satisfied by  $Q$ : only (2.4) is not immediate, but it follows from the prime number theorem.

Analogously to (2.12), we define

$$\begin{aligned} \tilde{S}(H) &:= \{h \in (0, H] : (\tilde{Q}(H), h) = 1 \text{ and } h \equiv a \pmod{q}\}, \\ \tilde{T}(H) &:= \{h \in (0, H] : (\tilde{Q}(H), h) = 1 \text{ and } h \not\equiv a \pmod{q}\}. \end{aligned} \quad (5.20)$$

Proposition 2.3 will follow from the next lemma.

**Lemma 5.5.** *Let  $H$  be a real parameter tending to infinity, and let  $\tilde{Q}(H)$  be as in (5.19). We have*

$$|\tilde{T}(H)| \ll \frac{H}{\log H}. \quad (5.21)$$

Moreover, there is a constant  $A = A(q)$ , depending on  $q$  at most, such that for all sufficiently large  $X$ , there is some  $H$  satisfying

$$\frac{X}{(\log X)^A} \leq H \leq X, \quad (5.22)$$

such that

$$|\tilde{S}(H)| \gg_q H \frac{\phi(\tilde{Q}(H))}{\tilde{Q}(H)}. \quad (5.23)$$

The implied constant in (5.21) is absolute, and that in (5.23) depends on  $q$  at most.



*Proof of Proposition 2.3.* Let  $S(H)$  and  $T(H)$  be as in (2.12). If  $p_0 \neq 1$  then by (2.1) there are at most  $H/p_0 < H/\log H$  multiples of  $p_0$  in  $T(H)$ , so

$$|T(H)| \ll \frac{H}{\log H}$$

by (5.21). We also have  $|S(H)| \geq |\tilde{S}(H)|$ . An application of Lemma 5.1 reveals that

$$\frac{\phi(\tilde{Q}(H))}{\tilde{Q}(H)} = \prod_{p \in \mathcal{P}(H)} \left(1 - \frac{1}{p}\right) \gg_q \begin{cases} \frac{1}{\log H} \left(\frac{\log H}{\log \log H}\right)^{1/\phi(q)} & \text{if } a \equiv 1 \pmod{q}, \\ \frac{1}{\log H} \left(\frac{\log t(H)}{\log \log H}\right)^{1/\phi(q)} & \text{if } a \not\equiv 1 \pmod{q}. \end{cases}$$

Therefore, in either case, combining (5.21) and (5.23) gives

$$|S(H)| - |T(H)| \gg |\tilde{S}(H)| - |\tilde{T}(H)| \gg_q H \frac{\phi(\tilde{Q}(H))}{\tilde{Q}(H)} \gg H \frac{\phi(Q(H))}{Q(H)}.$$

Proposition 2.3 now follows from Lemma 5.5.  $\square$

*Proof of Lemma 5.5.* We assume  $a \not\equiv 1 \pmod{q}$  as the case  $a \equiv 1 \pmod{q}$  is similar and simpler.

There are  $\ll H/\log H$  primes in  $\tilde{T}(H)$ , so let us count the composites  $h \in \tilde{T}(H)$ . If  $h = pm$  for some prime  $p > H/(\log H)^2$ , with  $m > 1$ , then  $m < (\log H)^2$  is composed only of primes  $> \log H$  and  $\equiv 1 \pmod{q}$ , by the construction of  $\mathcal{P}(H)$ . Thus,  $m$  must be prime itself, and  $p \leq H/\log H$ . We partition  $(H/(\log H)^2, H/\log H]$  into sub-intervals  $I_l = (e^{l-1}H/(\log H)^2, e^l H/(\log H)^2]$ , and  $(\log H, (\log H)^2]$  into sub-intervals  $J_l = (\log H, (\log H)^2/e^l]$ ,  $1 \leq l \leq \log \log H$ , and using the prime number theorem, we deduce that the contribution from elements with a large prime factor is at most

$$\sum_{1 \leq l \leq \log \log H} \sum_{\substack{p \in I_l \\ p \not\equiv 1 \pmod{q}}} \sum_{\substack{p' \in J_l \\ p' \equiv 1 \pmod{q}}} 1 \ll \sum_{1 \leq l \leq \log \log H} \frac{e^l H}{(\log H)^3} \frac{(\log H)^2}{e^l \log \log H} \ll \frac{H}{\log H}.$$

If  $h = pm$  with  $p \equiv a \pmod{q}$ , then  $p > H/t(H)$ , and  $m < t(H)$  must be composed only of primes  $\equiv 1 \pmod{q}$ , a contradiction as  $h \not\equiv a \pmod{q}$ . The only elements left uncounted must be composed only of primes  $p \equiv 1 \pmod{q}$  with  $\log H < p < t(H)$ . By (5.5), the number of such elements is at most

$$\Psi(H, t(H)) = H \exp(-u \log u - u \log \log u + O(u)),$$

where

$$u = \frac{\log H}{\log t(H)} = \frac{2 \log \log H}{\log \log \log H}.$$

Thus

$$u \log u + u \log \log u + O(u) \sim u \log u \sim 2 \log \log H,$$

and so

$$\Psi(H, t(H)) \ll \frac{H}{\log H}.$$

Combining these estimates yields (5.21).

Now suppose  $H$  is in the range (5.22). To bound the size of  $\tilde{S}(H)$  from below we will first do the same for

$$S'(X) := \{h \in (0, X] : (Q'(X), h) = 1 \text{ and } h \equiv a \pmod{q}\},$$

where

$$Q'(X) := q \prod_{p \in \mathcal{P}'(X)} p, \quad \mathcal{P}'(X) := \mathcal{P}(X) \setminus \{p \leq \log X : p \equiv 1 \pmod{q}\}.$$

Now  $pm \in S'(X)$  if  $X/t(X) < p \equiv a \pmod{q}$  and  $m \in \mathcal{S}(X/p)$ . We partition  $(X/t(X), X]$  into sub-intervals  $I_l = (e^{l-1}X/t(X), e^l X/t(X)]$ ,  $1 \leq l \leq \log t(X)$ , and deduce, using the prime number theorem for arithmetic progressions and Lemma 5.2, that

$$\begin{aligned} |S'(X)| &\geq \sum_{1 \leq l \leq \log t(X)} \sum_{\substack{p \in I_l \\ p \equiv a \pmod{q}}} \sum_{m \in \mathcal{S}(X/p)} 1 \\ &\gg_q \sum_{1 \leq l \leq \frac{1}{2} \log t(X)} \frac{e^l X}{t(X) \log X} \cdot \frac{t(X)}{e^l \log t(X)} (\log t(X))^{1/\phi(q)} \\ &\gg \frac{X}{\log X} (\log t(X))^{1/\phi(q)}. \end{aligned} \tag{5.24}$$

Now, we may write any  $h \in S'(X)$  uniquely as  $h = dm$ , where  $d$  is composed only of primes  $p \leq \log X$  with  $p \equiv 1 \pmod{q}$ , and  $m \in \tilde{S}(X)$ . Thus, by (5.24), there is a constant  $c_1(q) > 0$ , depending on  $q$  at most, such that for all sufficiently large  $X$ ,

$$c_1(q) \frac{X}{\log X} (\log t(X))^{1/\phi(q)} \leq |S'(X)| = \sum_{\substack{d \leq X \\ p|d \Rightarrow p \leq \log X \\ p \equiv 1 \pmod{q}}} \sum_{\substack{m \leq X/d \\ m \in \tilde{S}(X)}} 1 \leq \sum_{\substack{d \leq X \\ p|d \Rightarrow p \leq \log X \\ p \equiv 1 \pmod{q}}} |\tilde{S}(X/d)|. \tag{5.25}$$

The inequality on the right is not immediate: in fact if  $Z \leq X$ , then  $\tilde{S}(X) \cap (0, Z] \subseteq \tilde{S}(Z)$ . To see this, first note that as all of the functions used to define  $\mathcal{P}(X)$  are monotonically increasing with  $X$ ,

$$\mathcal{P}(Z) \subseteq \mathcal{P}(X) \cup \{t(Z) \leq p \leq t(X) : p \equiv 1 \pmod{q}\}.$$

Suppose  $m \in \tilde{S}(X) \cap (0, Z]$ , but  $m \notin \tilde{S}(Z)$ . Then  $p \in \mathcal{P}(Z)$  for some  $p \mid m$ , but  $p \notin \mathcal{P}(X)$ , so  $t(Z) \leq p \leq t(X)$  and  $p \equiv 1 \pmod{q}$ . Since  $m \equiv a \not\equiv 1 \pmod{q}$ , there must be some  $p' \mid m$  with  $p' \not\equiv 1 \pmod{q}$  and  $p' \leq m/p \leq Z/t(Z) \leq X/t(X)$ . Then  $p' \in \mathcal{P}(X)$ , a contradiction.

Suppose for a contradiction that for some constant  $c_2(q) > 0$ , depending on  $q$  at most, we have

$$|\tilde{S}(H)| \leq \frac{c_1(q)}{3c_2(q)} \frac{H}{\log X} \left( \frac{\log t(X)}{\log \log X} \right)^{1/\phi(q)} \quad (5.26)$$

for all  $H$  in the range (5.22). Then

$$\begin{aligned} \sum_{\substack{d \leq (\log X)^A \\ p|d \Rightarrow p \leq \log X \\ p \equiv 1 \pmod{q}}} |\tilde{S}(X/d)| &\leq \frac{c_1(q)}{3c_2(q)} \frac{X}{\log X} \left( \frac{\log t(X)}{\log \log X} \right)^{1/\phi(q)} \sum_{\substack{d \leq (\log X)^A \\ p|d \Rightarrow p \leq \log X \\ p \equiv 1 \pmod{q}}} \frac{1}{d} \\ &\leq \frac{c_1(q)}{3c_2(q)} \frac{X}{\log X} \left( \frac{\log t(X)}{\log \log X} \right)^{1/\phi(q)} \prod_{\substack{p \leq \log X \\ p \equiv 1 \pmod{q}}} \left( 1 - \frac{1}{p} \right)^{-1} \\ &\leq \frac{c_1(q)}{3} \frac{X}{\log X} (\log t(X))^{1/\phi(q)}, \end{aligned} \quad (5.27)$$

provided  $X$  is sufficiently large, and for a suitable choice of  $c_2(q)$  (given by Lemma 5.1).

Now, by the fundamental lemma of Brun's sieve, we have

$$|\tilde{S}(X/d)| \ll \frac{X}{d} \prod_{p \in \mathcal{P}(X/d)} \left( 1 - \frac{1}{p} \right) \quad (5.28)$$

for any  $d$ . If  $(\log X)^A < d \leq \sqrt{X}$ , then  $\log(X/d) \asymp \log X$ , and applying Lemma 5.1 to the sieve upper bound (5.28), we see that

$$\sum_{\substack{(\log X)^A < d \leq \sqrt{X} \\ p|d \Rightarrow p \leq \log X \\ p \equiv 1 \pmod{q}}} |\tilde{S}(X/d)| \leq c_3(q) \frac{X}{\log X} \left( \frac{\log t(X)}{\log \log X} \right)^{1/\phi(q)} \sum_{\substack{(\log X)^A < d \leq \sqrt{X} \\ p|d \Rightarrow p \leq \log X \\ p \equiv 1 \pmod{q}}} \frac{1}{d} \quad (5.29)$$

for some constant  $c_3(q) > 0$ . By lemmas 5.4 and 5.1 respectively, we have

$$\begin{aligned} \sum_{\substack{(\log X)^A < d \leq \sqrt{X} \\ p|d \Rightarrow p \leq \log X \\ p \equiv 1 \pmod{q}}} \frac{1}{d} &\leq \prod_{\substack{p \leq \log X \\ p \equiv 1 \pmod{q}}} \left( 1 - \frac{1}{p} \right)^{-1} (1 + o(1)) e^{-\gamma} \int_A^\infty \rho(v) dv \\ &\leq c_4(q) (\log \log X)^{1/\phi(q)} \int_A^\infty \rho(v) dv \end{aligned} \quad (5.30)$$

for some constant  $c_4(q) > 0$ . Now by (5.4),

$$\int_A^\infty \rho(v) dv \rightarrow 0 \quad \text{as } A \rightarrow \infty,$$

so we may choose  $A = A(c_1(q), c_3(q), c_4(q)) = A(q)$  so that

$$\int_A^\infty \rho(v) dv \leq \frac{c_1(q)}{4c_3(q)c_4(q)}.$$

For any such  $A$ , combining (5.29) and (5.30) yields

$$\sum_{\substack{(\log X)^A < d \leq \sqrt{X} \\ p|d \Rightarrow p \leq \log X \\ p \equiv 1 \pmod{q}}} |\tilde{S}(X/d)| \leq \frac{c_1(q)}{4} \frac{X}{\log X} (\log t(X))^{1/\phi(q)}. \quad (5.31)$$

Finally, using Rankin's trick, we see that

$$\begin{aligned} \sum_{\substack{\sqrt{X} < d \leq X \\ p|d \Rightarrow p \leq \log X \\ p \equiv 1 \pmod{q}}} |\tilde{S}(X/d)| &\leq \sum_{\substack{\sqrt{X} < d \leq X \\ p|d \Rightarrow p \leq \log X}} \frac{X}{d} \left( \frac{d}{\sqrt{X}} \right)^{1/3} \leq X^{5/6} \prod_{p \leq \log X} \left( 1 - \frac{1}{p^{2/3}} \right)^{-1} \\ &\leq X^{5/6} \exp \left( \sum_{p \leq \log X} \frac{3}{p^{2/3}} \right) \leq X^{5/6} \exp(9(\log X)^{1/3}) \\ &= X^{5/6+o(1)} \end{aligned} \quad (5.32)$$

by the prime number theorem.

Combining (5.25), (5.27), (5.31), and (5.32), we obtain  $c_1(q) \leq 2c_1(q)/3$ , which is absurd. We conclude that for all sufficiently large  $X$ , there is some  $H$  in the range (5.22) for which

$$|\tilde{S}(H)| \gg_q \frac{H}{\log X} \left( \frac{\log t(X)}{\log \log X} \right)^{1/\phi(q)} \gg \frac{H}{\log H} \left( \frac{\log t(H)}{\log \log H} \right)^{1/\phi(q)}.$$

A final application of Lemma 5.1 shows that this is  $\gg_q H\phi(\tilde{Q}(H))/\tilde{Q}(H)$ .  $\square$

## 6. A LOWER BOUND

In this section we will show how to obtain a quantitative version of Theorem 1.1. We will use the assumptions and notation of sections 3 – 5, and show that

$$|\{p_{r+1} \leq Y : p_{r+1} \equiv p_r \equiv a \pmod{q} \text{ and } p_{r+1} - p_r < \epsilon \log p_r\}| \geq Y^{1/3(\log \log Y)^A} \quad (6.1)$$

for all sufficiently large  $Y$ . Here  $A = A(q)$  is the constant given in Lemma 5.5. This lower bound could be improved by a sharpening of the range (5.22) for  $H$ .

We will first prove that the estimate

$$\sum_{N < n \leq 2N} \Lambda(n; \mathcal{H}, k + \ell)^4 \ll N(\log N)^{19k+4\ell} \quad (6.2)$$

holds, with an absolute implied constant. For by (4.1) and (4.2),

$$\begin{aligned}
\sum_{N < n \leq 2N} \Lambda(n; \mathcal{H}, k + \ell)^4 &= \sum'_{d_1, \dots, d_4} \lambda_{d_1} \cdots \lambda_{d_4} \sum_{\substack{N < n \leq 2N \\ [d_1, \dots, d_4] | P(n; \mathcal{H})}} 1 \\
&= \sum'_{d_1, \dots, d_4} \lambda_{d_1} \cdots \lambda_{d_4} \sum_{\substack{m \bmod [d_1, \dots, d_4] \\ \in \Omega([d_1, \dots, d_4])}} \sum_{\substack{N < n \leq 2N \\ n \equiv m \bmod [d_1, \dots, d_4]}} 1 \\
&\leq \sum_{\substack{d_1, \dots, d_4 \\ \text{squarefree}}} |\lambda_{d_1} \cdots \lambda_{d_4}| \sum_{\substack{m \bmod [d_1, \dots, d_4] \\ \in \Omega([d_1, \dots, d_4])}} \left( \frac{N}{[d_1, \dots, d_4]} + O(1) \right) \\
&\ll N(\log R)^{4(k+\ell)} \sum_{\substack{d_1, \dots, d_4 \leq R \\ \text{squarefree}}} \frac{|\Omega([d_1, \dots, d_4])|}{[d_1, \dots, d_4]}.
\end{aligned} \tag{6.3}$$

To see the last inequality, note that  $[d_1, \dots, d_4] \leq R^4 = N^{1-4\epsilon'} = o(N)$ , and so  $N/[d_1, \dots, d_4] + O(1) \ll N/[d_1, \dots, d_4]$ .

As observed in Section 4,  $|\Omega(d)| \leq k^{\omega(d)}$  for squarefree  $d$ , so

$$\begin{aligned}
\sum_{\substack{d_1, \dots, d_4 \leq R \\ \text{squarefree}}} \frac{|\Omega([d_1, \dots, d_4])|}{[d_1, \dots, d_4]} &\leq \sum_{D \leq R^4} \frac{\mu^2(D) k^{\omega(D)}}{D} \sum_{\substack{d_1, \dots, d_4 \\ [d_1, \dots, d_4] = D}} 1 \\
&= \sum_{D \leq R^4} \frac{\mu^2(D) (15k)^{\omega(D)}}{D} \leq \prod_{p \leq R^4} \left( 1 + \frac{15k}{p} \right) \\
&\ll (\log R^4)^{15k}.
\end{aligned} \tag{6.4}$$

Since  $R^4 < N$ , combining (6.3) and (6.4) yields (6.2).

Now choose  $N$  so that (3.2) holds. If we restrict the outer sum in the definition of  $\mathcal{L}$  to those  $n$  for which  $(Qn, Qn + H]$  contains a prime string  $p_{r+1} \equiv p_r \equiv a \pmod{q}$ , we remove no positive terms. Thus, if  $\sum^*$  denotes this restricted sum, then

$$\begin{aligned}
\mathcal{L} &\leq \\
&\frac{1}{N} \left( \frac{\phi(Q)}{Q} \right)^k \sum_{N < n \leq 2N}^* \left( \sum_{h \in S} \vartheta(Qn + h) - \sum_{h \in T} \vartheta(Qn + h) - \log 3QN \right) \Lambda_R(n; \mathcal{H}, k + \ell)^2.
\end{aligned} \tag{6.5}$$

For each  $n \in (N, 2N]$ ,

$$\sum_{h \in S} \vartheta(Qn + h) - \sum_{h \in T} \vartheta(Qn + h) - \log 3QN \leq H \log 3QN, \tag{6.6}$$

and by the Cauchy-Schwartz inequality,

$$\sum_{N < n \leq 2N}^* \Lambda_R(n; \mathcal{H}, k + \ell)^2 \leq \left( \sum_{N < n \leq 2N}^* 1 \right)^{1/2} \left( \sum_{N < n \leq 2N} \Lambda_R(n; \mathcal{H}, k + \ell)^4 \right)^{1/2}. \quad (6.7)$$

Combining (6.5) – (6.7) yields

$$\sum_{N < n \leq 2N}^* 1 \geq N^2 (Q/\phi(Q))^{2k} \mathcal{L}^2(H \log 3QN)^{-2} \left( \sum_{N < n \leq 2N} \Lambda_R(n; \mathcal{H}, k + \ell)^4 \right)^{-1}.$$

Using  $H = \epsilon \log N$ ,  $\log 3QN = (1 + o(1)) \log N$ , and  $Q/\phi(Q) \geq 1$ , then applying (3.2) and (6.2), we see that the right-hand side is  $\gg_{k,q} N/(\log N)^{17k+2}$ . Since  $k$  depends on  $\epsilon$ , we may write

$$\sum_{N < n \leq 2N}^* 1 \gg_{\epsilon,q} \frac{N}{(\log N)^{B(\epsilon)}}, \quad (6.8)$$

where  $B(\epsilon)$  is a constant depending on  $\epsilon$ .

Now fix a large number  $Y$ , and let

$$X := \epsilon \left( 1 + \frac{2c\epsilon}{(\log \log Y)^2} \right)^{-1} \log Y,$$

with  $c > 0$  fixed. By Lemma 5.5, we may choose  $H$  in the range

$$X/(\log X)^A \leq H \leq X$$

so that (3.2), hence (6.1), holds with  $N = \exp(H/\epsilon)$ . By (2.4),

$$3Q(H)N \leq \exp \left( \frac{H}{\epsilon} + \frac{cH}{(\log H)^2} \right) \leq Y,$$

because

$$\frac{H}{\epsilon} + \frac{cH}{(\log H)^2} = \frac{H}{\epsilon} \left( 1 + \frac{c\epsilon}{(\log H)^2} \right) \leq \frac{X}{\epsilon} \left( 1 + \frac{2c\epsilon}{(\log \log Y)^2} \right) = \log Y.$$

Here we have used  $\log H = (1 + o(1)) \log X = (1 + o(1)) \log \log Y$ . Also,

$$\log N = H/\epsilon \geq X/\epsilon (\log X)^A \geq \log Y/2 (\log \log Y)^A.$$

Therefore, using (6.8) as a lower bound for the number of prime strings up to  $Y$ , we deduce (6.1). (At best, we may have  $H = X$ , in which case we could deduce a lower bound of  $Y^{1-c'/( \log \log Y)^2}$ , for some constant  $c' > 0$ .)

## 7. CONCLUDING REMARKS

Proposition 2.2 is similar to a special case of Propositions 1 and 2 of [5], which are used to prove that

$$\liminf_{r \rightarrow \infty} \frac{p'_{r+\nu} - p'_r}{\phi(q) \log p'_r} \leq e^{-\gamma}(\sqrt{\nu} - 1)^2,$$

where  $p'_j$  denotes the  $j$ th smallest prime in the arithmetic progression  $a \bmod q$ ,  $(q, a) = 1$ . By considering  $H_\nu = (\nu - 1 + \epsilon) \log N$  instead of  $H$ ,  $Q = Q(H_\nu)$  instead of  $Q(H)$ , and

$$\mathcal{L}_\nu :=$$

$$\frac{1}{N} \left( \frac{\phi(Q)}{Q} \right)^k \sum_{N < n \leq 2N} \left( \sum_{h \in S} \vartheta(Qn + h) - \nu \sum_{h \in T} \vartheta(Qn + h) - \nu \log 3QN \right) \Lambda_R(n; \mathcal{H}, k + \ell)^2$$

instead of  $\mathcal{L}$ , it is possible to prove that the interval  $(Qn, Qn + H_\nu]$  contains a string of  $\nu + 1$  consecutive primes  $\equiv a \bmod q$ , for some  $n \in (N, 2N]$  and a sequence  $N \rightarrow \infty$ . It may be feasible to prove a similar result with  $H_\nu = (e^{-\gamma}(\sqrt{\nu} - 1)^2 + \epsilon) \log N$ .

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